MATH 7382: TOPICS IN PROBABILITY
PROBLEM SET 2

DUE: 10/16/12

(* means harder optional exercises)

1. (a) Show that the finite-dimensional unitary group[1] $U(N)$ is compact in the usual topology.

(b) Show that the infinite-dimensional unitary group $U(\infty) = \bigcup_{N=1}^{\infty} U(N)$ with the inductive limit topology (= natural topology of convergence of all matrix elements) is not compact.

(c)* Show that $U(\infty)$ is not locally compact.

2. In the algebra of Laurent polynomials $\mathbb{C}\left[u_1^{\pm 1}, \ldots, u_N^{\pm 1}\right]$ (as well as in the subalgebra of symmetric polynomials $LSym(N)$) consider the automorphism

$$u_i \mapsto u_i^{-1}, \qquad \text{for all } i = 1, \ldots, N.$$

Show that $s_\lambda(u^{-1}) = s_{\lambda^*}(u)$, where

$$\lambda^* = (-\lambda_N, -\lambda_{N-1}, \ldots, -\lambda_2, -\lambda_1).$$

3. [elementary and complete homogeneous symmetric polynomials] Let $e_k(u_1, \ldots, u_N)$, $k = 1, \ldots, N$, be the kth elementary symmetric polynomial: $e_k(u_1, \ldots, u_N) = \sum\limits_{1 \le i_1 < \ldots < i_k \le N} u_{i_1} \ldots u_{i_k}$.

Also, let $h_k(u_1, \ldots, u_N)$, $k = 1, 2, \ldots$, be the kth complete homogeneous symmetric polynomial: $h_k(u_1, \ldots, u_N) = \sum\limits_{1 \le i_1 \le \ldots \le i_k \le N} u_{i_1} \ldots u_{i_k}$.

(a) Show that $e_k(u_1, \ldots, u_N) = s_{(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)}(u_1, \ldots, u_N)$;

(b) Show that $h_k(u_1, \ldots, u_N) = s_{(k,0,\ldots,0)}(u_1, \ldots, u_N)$.

Hint: You may use the branching rule for the Schur polynomials in one of the two ways. (1) Write $e_k(u_1, \ldots, u_{N-1}, 1)$ in terms of e_j's in variables $(u_1, \ldots, u_{N-1})$; and same for h_k's. Compare this with the branching rule for Schur polynomials. (2) Use combinatorial formula for Schur polynomials directly.

4. [Schur polynomials in two variables; cf. Problem 2 in the first Problem Set] Let $N = 2$, and consider the Schur polynomials in two variables:

$$s_{(a,b)}(x, y) = \frac{x^{a+1} y^b - x^b y^{a+1}}{x - y}, \quad a \ge b \ge 0.$$

Write this Schur polynomial as a linear combination of the symmetrized monomials

$$m_{(c,d)}(x, y) := \begin{cases} x^c y^d + x^d y^c, & c > d \ge 0; \\ x^c y^c, & c = d \ge 0. \end{cases}$$

Explanation: In this way you get the Kostka numbers $K_{\lambda\mu}$ which are defined as

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

for all λ's and μ's with two parts.

[1]Which consists of all complex $N \times N$ matrices U with condition $UU^* = E$; here E is the identity matrix and U^* means transposed complex conjugate matrix.

5. Show that the number $\dim\lambda$ of standard Young tableaux of shape λ (λ is a nonnegative signature)[2] is equal to the coefficient by $x_1^{\lambda_1+N-1}x_2^{\lambda_2+N-2}\ldots x_N^{\lambda_N}$ in the polynomial

$$(x_1+\ldots+x_N)^N\prod_{1\le i<j\le N}(x_i-x_j),$$

where N is large enough ($N\ge\ell(\lambda)$).

Hint: use the corollary of the Pieri formula: $p_1^K=\sum_{\lambda\in\mathbb{GT}^+\colon|\lambda|=K}\dim\lambda\cdot s_\lambda$, and multiply by the Vandermonde.

6. [Plancherel specialization of the algebra of symmetric functions] Let us consider a specialization of the algebra of symmetric functions $Sym=\mathbb{R}[p_1,p_2,\ldots]$ (i.e. algebra morphism $Sym\to\mathbb{R}$) defined on the multiplicative generators (power sums) p_k as

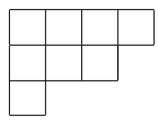
$$p_1\to1,\qquad p_2,p_3,p_4,\ldots\to0.$$

(a) Show that under this specialization e_k and h_k map to $1/k!$. **Hint:** use relations between the generating functions of p's, e's and h's.

(b) Show that s_λ maps to $\det\left[\dfrac{1}{(\lambda_i-i+j)!}\right]_{i,j=1}^N$ (for any $N\ge\ell(\lambda)$). **Hint:** use Jacobi-Trudi identity.

(c)* In fact, the image of s_λ is equal to $\dim\lambda/|\lambda|!$.

7. Let λ be represented by a Young diagram (where λ is a nonnegative signature). For example, if $\lambda=(4,3,1)$, then the corresponding diagram looks as



Let d be the number of boxes on the diagonal of λ ($d=2$ in the above example), and let n be the total number of boxes in the diagram ($n=8$ in the above example). Show that $d\le\sqrt{n}$.

[2] Recall also that $\dim\lambda$ is the dimension of the irreducible representation of $\mathfrak{S}(|\lambda|)$ corresponding to λ.