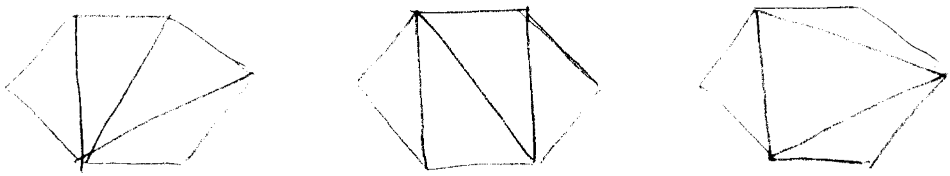
COMBINATORICS. PROBLEM SET 10.
ENUMERATION OF TREES

Seminar problems

Problem 10.1. Find the number of non-directed graphs on n vertices with simple edges and no loops.

Problem 10.2. What is the number of rooted plane trees with $n+1$ vertices?

Problem 10.3. Consider the following three triangulations of the hexagon:



Assume we are rotating the hexagon by $60°$, $120°$, ... How many other different triangulations each picture produces?

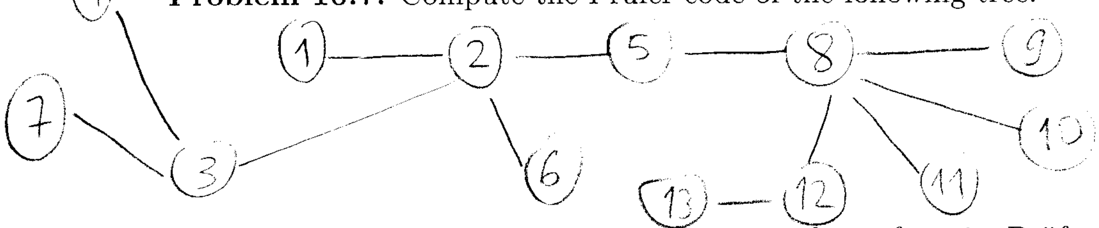
Problem 10.4. How many triangulations of the $(n+2)$-gon with labeled vertices are? What about the $(n+2)$-gon with unlabeled vertices? Let the latter number of triangulations be R_n. Give estimates for R_n of the form $a_n \leq R_n \leq b_n$, such that b_n/a_n has a polynomial asymptotics.

Problem 10.5. Give a recursive definition of

*) a rooted tree
*) a labeled rooted tree
*) a plane tree
*) a labeled plane tree

Problem 10.6 (Cayley's formula). Let T_n be the number of labeled rooted trees on n vertices, and $T(x) := \sum_{n\geq 1} T_n x^n/n!$. Show that $T(x) = xe^{T(x)}$. Applying the Lagrange inversion, show that $T_n = n^{n-1}$.

Problem 10.7. Compute the Prüfer code of the following tree:



Problem 10.8. Construct the labeled unrooted tree from its Prüfer code $(3,1,5,3,5,3,2,1,10)$.

Problem 10.9. Obtain Problem 10.6 using Prüfer codes.

Problem 10.10. From Problem 10.6 deduce the number of unrooted labeled trees on n vertices.

Problem 10.11.

Problem 10.12. Generalize the consruction of a Prüfer code for labeled rooted forests.

Homework

Problem 10.13 (*1*). Find the number of non-directed graphs on n vertices with simple edges (loops allowed).

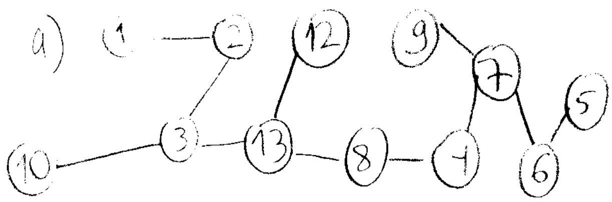
Problem 10.14 (*1*). Find the number of directed graphs on n vertices with simple edges (loops allowed).

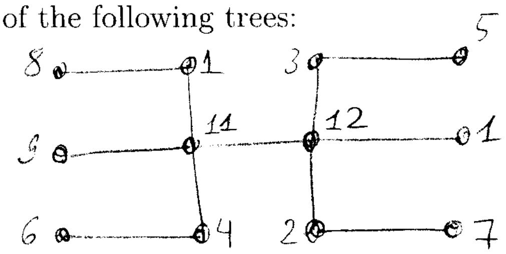
Problem 10.15 (*1*). What is the number of rooted plane trees with $n+2$ vertices such that the degree of the root is one?

Problem 10.16 (*1*). How many different cyclic orders are on n elements? (e.g., for $n = 3$ there are two such orders: 123 and 132).

Problem 10.17 (*1*). Draw all possible (unlabeled, unrooted) trees on n vertices, $n = 1, 2, 3, 4, 5, 6$.

Problem 10.18 (*2*). Compute the Prüfer code of the following trees:





Problem 10.19. Construct the labeled unrooted trees from their Prüfer codes: a) (*1*) $(1, 2, 3, 4, 5, 6, 1, 2, 3)$, b) (*1*) $(3, 5, 3, 5, 3, 5, 3, 5)$.

Problem 10.20 (*3*). Use the Prüfer code construction to show that the number of labeled unrooted trees on n vertices with degrees of vertices $d_1, \dots, d_n$ (d_j is the degree of the jth vertex) is $\frac{(n-2)!}{(d_1-1)!\dots(d_n-1)!}$. (Hint: think of what Prüfer sequences can occur from such trees.)

Problem 10.21 (*3*). Consider *bipartite trees* (labeled unrooted), i.e., trees in which vertices are of two types: black vertices with labels $1, \dots, n_b$, and white vertices with labels $n_b+1, \dots, n_b+n_w$. Each edge must connect a white and a black vertex. Show that the number of such bipartite trees is $n_w^{n_b-1} n_b^{n_w-1}$. (Argue similarly to the previous problem.)

Problem 10.22 (*3*). Take a rooted labeled forest on n vertices. Take roots of each component and connect them to a new vertex with label $n+1$. Observe that this will be an unrooted labeled tree on $n+1$ vertices. Deduce that the number of rooted labeled forests on n vertices is $(n+1)^{n-1}$.

Supplementary problem

Problem 10.23 (*5*). Deduce that the number of labeled rooted forests on n vertices with k components such that the root of the ith component has label i, is kn^{n-k-1} (use Problem 10.12).