COMBINATORICS. PROBLEM SET 11
EXPONENTIAL FORMULA

Seminar problems

Problem 11.1 (Exponential formula)**.** Let $f\colon \{1,2,\dots\}\to\mathbb{C}$ and $g\colon\{0,1,2,\dots\}\to\mathbb{C}$ be two functions, and $g(0)=1$. Let $F_e(x) := \sum_{j=1}^{\infty} f(j)x^j/j!$ and $G_e(x) := \sum_{k=0}^{\infty} g(k)x^k/k!$ be the corresponding *exponential* generating functions. Define a new function $h\colon\{0,1,2,\dots\}\to\mathbb{C}$ by

$$h(n) := \sum_{\pi=\{\pi_1,\dots,\pi_k\}} f(\#\pi_1)\dots f(\#\pi_k)g(k) \qquad (n\ge 1), \qquad h(0):=1, \tag{11.1}$$

where the sum is taken over all *partitions* π *of the set* $\{1,\dots,n\}$. Show that the exponential generating function for h is the composition $H_e(x) = G_e(F_e(x))$.

Problem 11.2. Show that the number of ways to divide n men into several groups, then order them in every group in a line, and finally order the groups in cyclic order is $(2^n-1)(n-1)!$.

Homework/seminar problems

Problem 11.3 (*1*)**.** Why does the composition $G_e(F_e(x))$ of generating functions in Problem 11.1 make sense?

Problem 11.4 (*3*)**.** Let $g\equiv 1$. Give a combinatorial interpretation of Problem 11.1 What combinatorial interpretation arises when also $f\equiv 1$?

Problem 11.5 (*3*)**.** Give a purely combinatorial proof in Problem 11.2.

Problem 11.6 (*2*)**.** Compute the exponential generating function for the number of labeled unrooted forests on n vertices such that every component of the forest is a linear (labeled unrooted) tree (i.e., tree of the form $i_1-i_2-i_3-\dots-i_k$).

Problem 11.7 (*2*)**.** Let $t(n)$ be the number of labeled rooted trees on n vertices, and let $T_e(x)=\sum_{n=1}^{\infty} t(n)x^n/n!$ be the corresponding exponential generating function. By cutting out the root, show that $t(n)$'s satisfy an identity of the form (11.1), and using the exponential formula, rederive the known identity $T_e(x) = xe^{T_e(x)}$.

Problem 11.8 (*2*)**.** Let $f(n)$ be the total number of (rooted labeled) forests on n vertices. Using the generating function $T_e(x)$ for $t(n)$ (the number of trees), find the exponential generating function for $f(n)$ (in terms of $T_e(x)$), and then find $f(n)$.

Problem 11.9 (*4*)**.** Let $f_k(n)$ denote the number of forests as in the previous problem with exactly k components. Find the two-variable generating function $F(x,t)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_k(n)t^kx^n/n!$.

Problem 11.10 (*2*)**.** Let $\hat{f}(n)$ be the number of (rooted labeled) forests on n vertices such that their connected components (i.e., trees) are linearly ordered. Find the exponential generating function for $\hat{f}(n)$.

Problem 11.11 (*3*)**.** Let $B(n)$ $(n\ge 1)$ be the nth *Bell number*, i.e., the number of partitions of the set $\{1,\dots,n\}$. Show that $\sum_{n=1}^{\infty} B(n)x^n/n! = \exp(e^x-1)$.