

# COMBINATORICS

## PROBLEM SET 5

### GENERATING FUNCTIONS AND ASYMPTOTICS

#### HOMEWORK

(5.1) Let  $a_0, a_1, a_2, \dots$  and  $b_0, b_1, b_2, \dots$  be two sequences. We say that these sequences  $\{a_n\}$  and  $\{b_n\}$  are *asymptotically equivalent* (denoted by  $a_n \sim b_n$ ) as  $n \rightarrow +\infty$ , if the limit  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and is equal to one. We say that  $\{a_n\}$  *grows slower* than  $\{b_n\}$  ( $= \{b_n\}$  *grows faster* than  $\{a_n\}$ ) if the same limit  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and is equal to zero.

“Find the asymptotics of a sequence  $\{a_n\}$ ” means to find some “known”  $\{b_n\}$  which is asymptotically equivalent to  $\{a_n\}$ . For example, the asymptotics of  $n^2 + \frac{3n}{7}$  is the same as of  $n^2$ . Another example: the asymptotics of  $\binom{n}{3}$  is the same as of  $\frac{1}{6}n^3$ .

There is a number of pattern sequences: the most ordinary monotonous sequences with simple well-known behaviour. These are (in decreasing rate of their growth):

$$n!; \quad a^n \quad (a > 1); \quad n^\alpha \quad (\alpha > 0); \quad \log n; \quad n^\alpha \quad (\alpha \leq 0); \quad a^n \quad (0 < a < 1).$$

Let  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ . The radius of convergence  $R$  of the series  $a_0 + a_1x + a_2x^2 + \dots$  is given by the formula  $\frac{1}{R} = \limsup_{n \rightarrow +\infty} \sqrt[n]{|a_n|}$ .

**Problem 5.1.** Find the asymptotics of the sequences: (a) (1)  $\binom{n}{k}$ , where  $k$  is fixed and  $n \rightarrow +\infty$ ; (b) (2)  $\sqrt{n^2 + 3n} - \sqrt{n^2 + 3n + 7}$ ; (c) (2)  $\binom{n}{k+1} - \binom{n}{k}$ , where  $k \geq 0$  is fixed and  $n \rightarrow +\infty$ .

**Problem 5.2.** Let the radius of convergence of the series  $a_0 + a_1x + a_2x^2 + \dots$  be  $R$ . What is the radius of convergence of (a) (1)  $a_0 + a_1x^3 + a_2x^6 + a_3x^9 + a_4x^{12} + \dots$ ; (b) (1)  $a_3 + a_4x + a_5x^2 + a_6x^3 + \dots$ ?

**Problem 5.3.** (1) Find the radius of convergence of the series  $\sum_{n=0}^{\infty} x^{(10^n)}$ .

**Problem 5.4.** (1) Give an example of a power series  $a_0 + a_1x + a_2x^2 + \dots$  which has radius of convergence  $R = \pi$ .

**Problem 5.5.** Give an example of a power series  $a_0 + a_1x + a_2x^2 + \dots$  with radius of convergence  $R = 1$  such that: (a) (1) the series converges for all  $x$  with  $|x| = 1$ ; (b) (1) the series diverges for any  $x$  with  $|x| = 1$ ; (c) (2) for some of  $x$ 's with  $|x| = 1$  the series converges, and for some of  $x$ 's with  $|x| = 1$  the series diverges.

(5.2) Let  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ . It is a property that the radius of convergence  $R$  of the series  $a_0 + a_1x + a_2x^2 + \dots$  is the distance from 0 to the nearest *singular point* of the function  $f$ .

**Problem 5.6.** Find the singular points of the functions and find the radii of convergence of the corresponding power series (= the distance from 0 to the closest singular point): (a) (1)  $f(z) = \sqrt{1-z}$ ; (b) (1)  $f(z) = \frac{1}{1-4z}$ ; (c) (2)  $f(z) = (1+z)^\alpha$  where  $\alpha \in \mathbb{R}$  is fixed (note the special case when  $\alpha$  is a positive integer); (d) (2)  $\frac{z}{e^z - 1}$ ; (e) (2)  $\frac{e^z - 1}{z}$ .

(5.3) If the series  $a_0 + a_1x + a_2x^2 + \dots$  converges for some  $x \in \mathbb{C}$  with  $|x| = r \in (0, +\infty)$ , then the sequence  $\{a_n\}$  grows slower than  $(\frac{1}{r} + \epsilon)^n$  for any  $\epsilon > 0$ . If  $R \in (0, +\infty)$  is the radius of convergence of the series  $a_0 + a_1x + a_2x^2 + \dots$ , then the sequence  $\{a_n\}$  grows slower than  $(\frac{1}{R} + \epsilon)^n$  for any  $\epsilon > 0$ . If the series  $a_0 + a_1x + a_2x^2 + \dots$  converges for all  $x \in \mathbb{C}$ , then the coefficients  $\{a_n\}$  grow slower than  $\epsilon^n$  for any  $\epsilon > 0$ .

**Problem 5.7.** By looking at the generating function of the Catalan numbers  $C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$ , find: (a) (2) singular points of  $C(x)$ ; (b) (1) the radius of convergence of the series  $\sum_{n=0}^{\infty} C_n x^n$ ; (c) (2) the exponential-type asymptotics (as in (5.3) above<sup>1</sup>) of the Catalan numbers  $C_n$  as  $n \rightarrow +\infty$ .

**Problem 5.8.** The generating function for the Fibonacci numbers is  $F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$ . Find: (a) (2) singular points of  $F(x)$ ; (b) (1) the radius of convergence of the series  $\sum_{n=0}^{\infty} F_n x^n$ ; (c) (2) the exponential-type asymptotics (as in (5.3) above) of the Fibonacci numbers  $F_n$  as  $n \rightarrow +\infty$ .

**Problem 5.9.** (3) Recall the explicit formula the Fibonacci numbers  $F_n$  (or derive it again by expanding the generating function  $F(x)$  as a sum of partial fractions), and find the *exact asymptotics* (not in the form of (5.3)) for the Fibonacci numbers (i.e., find  $c, d$  such that  $F_n \sim c \cdot d^n$ ).

(5.4) (*asymptotics of hypergeometric sequences*) A sequence  $\{a_n\}$  is called *hypergeometric*, if the ratio  $\frac{a_{n+1}}{a_n}$  is a rational function in  $n$ .

If for all sufficiently large  $n$  we have  $\frac{a_{n+1}}{a_n} = A \frac{n^k + \alpha_1 n^{k-1} + \dots + \alpha_k}{n^k + \beta_1 n^{k-1} + \dots + \beta_k}$  with  $\alpha_1 \neq \beta_1$ , then the sequence  $\{a_n\}$  grows as  $a_n \sim c A^n n^{\alpha_1 - \beta_1}$  for some constant  $c > 0$ .

**Problem 5.10.** (a) (1) Compute  $\frac{C_{n+1}}{C_n}$ , where  $C_n$  is the  $n$ th Catalan number; (b) (1) Using this, find the asymptotics of  $C_n$  as in (5.4) above.<sup>2</sup>

**Problem 5.11.** (a) (2) Find the power series expansion of the function  $f(x) = (a - x)^\alpha$  (where  $a, \alpha \in \mathbb{R}$  are fixed), that is,  $f(x) = a_0 + a_1x + a_2x^2 + \dots$ . (b) (2) Compute  $\frac{a_{n+1}}{a_n}$ . (c) (2) Find the asymptotics of  $a_n$  as in (5.4) above.

**Problem 5.12.** Let  $k$  be a fixed positive integer, and let  $a_n := \frac{k}{2n+k} \binom{2n+k}{n}$  (note that for  $k = 1$  these are the Catalan numbers; for general  $k$  these numbers are called the *ballot numbers*). (a) (2) Compute  $\frac{a_{n+1}}{a_n}$ . (b) (2) Find the asymptotics of  $a_n$  as in (5.4) above.

(5.5) *Stirling formula* for the asymptotics of factorial:  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$  as  $n \rightarrow +\infty$ .

**Problem 5.13.** Using Stirling formula, find the *exact asymptotics* (not in the form of (5.3) or (5.4)) of the following sequences as  $n \rightarrow +\infty$ : (a) (2)  $\binom{3n}{n}$ ; (b) (2)  $\binom{6n}{2n} - \binom{3n}{n}$ .

**Problem 5.14.** (2) Use Stirling formula and the explicit formula for the Catalan numbers  $C_n$  to derive the *exact asymptotics* (not in the form of (5.3) or (5.4)) of  $C_n$ .

#### SUPPLEMENTARY PROBLEMS

**Problem 5.15.** (4) Recall the generating function for the Motzkin numbers, find its singular points, the radius of convergence of the corresponding power series and the asymptotics in the form (5.3) of the Motzkin numbers  $M_n$  as  $n \rightarrow +\infty$ .

<sup>1</sup>That is, find  $R$  such that the sequence  $C_n$  grows slower than  $(\frac{1}{R} + \epsilon)^n$  for any  $\epsilon > 0$ .

<sup>2</sup>That is, find real constants  $A$  and  $s$  such that  $C_n \sim c A^n n^s$  for some positive constant  $c$ .