COMBINATORICS. PROBLEM SET 8. DIRICHLET GF'S

Seminar problems

Problem 8.1. Find a quick way of writing out all prime numbers from 2 to a given number N.

$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the *Riemann zeta function*.

Problem 8.2. Show that $\zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$.

Let $f\colon \mathbb{N} \to \mathbb{C}$ be a function of a natural number (such functions are called *arithmetic*). (Equivalently, we can speak of the sequence $\{f(1), f(2), f(3), \dots\}$.) Associate to f the *Dirichlet GF*: $A(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$.

Problem 8.3. Let $A(s)$ and $B(s)$ be the Dirichlet GF's for two arithmetic functions f and g, respectively. Find (in terms of f and g) the sequence whose Dirichlet GF is $A(s)B(s)$.

Problem 8.4. Let for $n \in \mathbb{N}$ by $d(n)$ denote the number of divisors of n. E.g., $d(6) = 4$. Find the Dirichlet GF for $d(n)$.

An arithmetic function f is called *multiplicative* iff $f(ab) = f(a)f(b)$ for any relatively prime a, b.

Problem 8.5. Show that a multiplicative arithmetic function is completely defined by its values on powers of prime numbers.

Problem 8.6. Show that and arithmetic function f is multiplicative iff the corresponding Dirichlet GF can be represented in the following form: $\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots)$. (In fact, this identity for multiplicative arithmetic functions is useful in many of the homework problems.)

Problem 8.7. Let $\mu(n)$ be the arithmetic function with Dirichlet GF which is the inverse of the Riemann zeta function: $\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} (1 - p^{-s})$. Find a $\mu(n)$ for all n in an explicit form.

Problem 8.8. Use the inclusion-exclusion principle to show that $\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) = 1$.

Problem 8.9. Use Dirichlet GF's to prove the *Möbius inversion theorem*:

$$f(n) = \sum_{d\colon d|n} g(d) \qquad \Leftrightarrow \qquad g(n) = \sum_{d\colon d|n} \mu(\tfrac{n}{d}) f(d).$$

for any arithmetic functions f and g. (In fact, this theorem is useful in many of the homework problems.)

Problem 8.10. Prove the inclusion-exclusion principle: B is a finite set, each element can possess or not possess some of the properties $c_1, \dots, c_m$. Let $N(c_{i_1}, \dots, c_{i_k})$ be the number of elements each of which possesses the properties $c_{i_1}, \dots, c_{i_k}$. Let also $N(1) = \#B$. Then the number of elements in B which do not possess any of the properties $c_1, \dots, c_m$, is

$$N(1) - N(c_1) - \dots - N(c_m) + N(c_1, c_2) + \dots + N(c_{m-1}, c_m) - N(c_1, c_2, c_3) - \dots.$$

Problem 8.11. Show that the Dirichlet GF's of nonzero multiplicative arithmetic functions form a group with respect to the multiplication of Dirichlet GF's.

Problem 8.12. A ticket has a 6-digit number $abcdef$. Ticket is *lucky* iff $a+b+c=d+e+f$. Find (using the inclusion-exclusion principle) the total number of lucky tickets. (**Hint** 1: by using the bijection $abcdef \to abc(9-d)(9-e)(9-f)$, conclude that the number of lucky tickets is the same as the number of tickets with sum of the digits equal to 27. **Hint 2**: assume that $a, b, c, d, e, f \geq 0$ and use the inclusion-exclusion principle with c_i being the property that the ith number in a ticket is ≥ 10.)

Problem 8.13. A *disorder* on the set $\{1,\ldots,n\}$ is a permutation σ of $\{1,\ldots,n\}$ such that $\sigma(k) \neq k$ for any k. Find (using the inclusion-exclusion principle) the number of disorders on $\{1,\ldots,n\}$. (**Hint**: c_i is the property of a permutation to fix i, i.e., $\sigma(i) = i$.)

Homework

Problem 8.14. Use the inclusion-exclusion principle to show that

$$\max(a_1,\ldots,a_n) = a_1+\cdots+a_n-\min(a_1,a_2)-\cdots-\min(a_{n-1},a_n)+\min(a_1,a_2,a_3)+\cdots+(-1)^n\min(a_1,\ldots,a_n).$$

(**Specify the properties** c_i **explicitly**.)

Problem 8.15. Let $\varphi(n)$ be the number of numbers among $\{1,\ldots,n-1\}$ which are relatively prime to n. Show that for $n = p_1^{k_1}\ldots p_m^{k_m}$ (p_i distinct primes) we have $\varphi(n) = n(1-\frac{1}{p_1})\ldots(1-\frac{1}{p_m})$. Use the inclusion-exclusion principle. (**Specify the properties** c_i **explicitly**.)

Problem 8.16. From the above problem, show that the arithmetic function φ is multiplicative.

Problem 8.17. Find the Dirichlet GF for φ.

Problem 8.18. How many of the following n^2 fractions

$$\begin{array}{ccccc} 1/1 & 1/2 & 1/3 & \ldots & 1/n \\ 2/1 & 2/2 & 2/3 & \ldots & 2/n \\ \multicolumn{5}{c}{\dotfill} \\ n/1 & n/2 & n/3 & \ldots & n/n \end{array}$$

are irreducible? Use the inclusion-exclusion principle. (**Specify the properties** c_i **explicitly**.)

Problem 8.19. Let $\sigma(n)$ be the sum of all the divisors of n (n included), e.g., $\sigma(6) = 12$. Show that this is a multiplicative arithmetic function.

Problem 8.20. Find the Dirichlet GF for the arithmetic function σ.

Problem 8.21. Show that $\sum_{\delta:\, \delta|n} \mu(\delta)d(\frac{n}{\delta}) = 1$ for $n \geq 1$. (Here μ and d are defined above.)