COMBINATORICS. FINAL EXAM

Problem 1. Let $F(u) = \sum_{k=0}^{\infty} \frac{(k+1)^2}{u^k}$. Find the similar expansion of $F(u+1)$ as the sum over negative powers of u, i.e., find the c_k's in $F(u+1) = \sum_{k=0}^{\infty} \frac{c_k}{u^k}$.

Problem 2. Solve the recurrence relation (and check yourself by substituting your answer back in the relation):

$$a_{n+2} = 4a_n + 2^n \quad (n \geq 0), \qquad a_0 = 0, \quad a_1 = \tfrac{1}{4}.$$

Problem 3. Recall that the number of standard Young tableaux (= the ways to grow a Young diagram by adding one box at a time) of shape (n, n) is C_n, the nth Catalan number. E.g., for $n = 3$ we have 5 such tableaux:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 6 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array} \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & 6 \end{array} \qquad \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & 6 \end{array}$$

Compute the number of standard Young tableaux of shape $(n+2, n)$ (you can use the hook formula only to check your answer). E.g., for $n = 2$ one has 9 such tableaux:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 4 & 6 & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 4 & 5 \\ 3 & 6 & & \end{array}$$

$$\begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array} \qquad \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & 6 & & \end{array} \qquad \begin{array}{cccc} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{array} \qquad \begin{array}{cccc} 1 & 3 & 4 & 6 \\ 2 & 5 & & \end{array}$$

Problem 4. What is the number of standard Young tableaux of shape $(n+1, n)$?

Problem 5. Express the number of partitions of 100 into even parts which are ≥ 3 and ≤ 10 as a suitable coefficient in a certain generating function.

Problem 6. An arithmetic function f is called *strongly multiplicative* if $f(mn) = f(m)f(n)$ for all m, n (not necessary relatively prime). Find a strongly multiplicative function $a(n)$ such that the arithmetic function $b(n) = \sum_{d:\, d|n} a(n)$ is not strongly multiplicative. Will $b(n)$ be just multiplicative (i.e., $b(xy) = b(x)b(y)$ for relatively prime x, y)?

Problem 7. Let $\mu(n)$ be the Möbius arithmetic function. Find (in terms of the zeta function $\zeta(s)$) the Dirichlet GF for the sequence $\{|\mu(n)|\}$.

Problem 8. Let $d(n)$ be the number of divisors of n, and let

$$a(n) := \sum_{k:\, k|n} \left(\sum_{m:\, m|k} d(m) \right).$$

Find the Dirichlet GF for the sequence $a(n)$.

Problem 9. Consider the complete (labeled) graph $K_{1,1,5}$ which has two distinguished vertices A, B connected to each other, and 5 other vertices $C_1, \ldots, C_5$ of degree 2 which are connected to both A and B. Compute the number of spanning trees for this graph (in the labeled setting, as usual).

Problem 10. Consider the model of gluing of edges of a $2n$-polygon. Show that the genus of the resulting surface does not exceed $\left[\frac{n}{2}\right]$ (integer part).

Problem 11. Compute the number of gluings of the hexagon ($n = 3$) which give the torus.

Supplementary problems

Problem 12. Express the number of integer solutions of the equation

$$a + b + c = 100$$

with constraints $60 \geq a \geq b \geq c \geq 4$ as a suitable coefficient in a certain generating function. (Hint: you will need a two-variable generating function.)

Problem 13. Let λ be the multiplicative function such that $\lambda(p^k) = (-1)^k$ (p prime). Prove that

$$\sum_{d\colon\, d|n} \lambda(d) = \begin{cases} 1, & n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 14. Solve Problem 9 for $K_{1,1,n}$.

Problem 15. Compute the number of gluings of the octagon ($n = 4$) which give the torus.