

COMBINATORICS. MIDTERM TEST. SOLUTIONS

1. BASICS AND GF'S

Problem 1. Find $[x^{3n}]e^{2x}$ (the coefficient by x^{3n} in the power series expansion of e^{2x}).

Solution 1. Since

$$e^{2x} = \sum_{k=0}^{\infty} \frac{2^k x^k}{k!},$$

we have

$$[x^{3n}]e^{2x} = \{\text{take } k = 3n \text{ above}\} = \frac{2^{3n}}{(3n)!}.$$

Problem 2. Show that

$$(1) \quad \binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \dots + 2^k \binom{n}{k} + \dots + 2^n \binom{n}{n} = 3^n.$$

Solution 2. Take the binomial theorem $(1+x)^n = \sum_{k=0}^n x^k \binom{n}{k}$, and set $x = 2$. See also Problem 11 below.

Problem 3. What is the total number of ways in which the letters in the word “PETERSBURG” can be rearranged?

Solution 3. The answer is $\frac{10!}{2! \cdot 2!}$. If all letters were distinct (“PE₁TE₂R₁SBUR₂G”), it would be 10! rearrangements, but to any rearrangement of distinct letters correspond $2! \cdot 2!$ rearrangements of the original letters, because we can interchange E₁ with E₂, as well as R₁ with R₂.

Problem 4. Solve the recurrence (using generating functions):

$$a_{n+2} = 3a_{n+1} + 4a_n - 6n - 1 \quad (n \geq 0), \quad a_1 = 4, \quad a_0 = 2.$$

(Advice: substitute your answer into the equation to check yourself.)

Solution 4. Multiply both sides of the recurrence relation by x^{n+2} and sum over all $n \geq 0$ (the range where the relation holds):

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = 3x \sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 4x^2 \sum_{n=0}^{\infty} a_n x^n - 6 \sum_{n=0}^{\infty} n x^{n+2} - \sum_{n=0}^{\infty} x^{n+2}.$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of a_n 's, we have

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = f(x) - a_0 - a_1 x = f(x) - 2 - 4x, \quad \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = f(x) - a_0 = f(x) - 2, \quad \sum_{n=0}^{\infty} a_n x^n = f(x).$$

Moreover, we have

$$\sum_{n=0}^{\infty} n x^{n+2} = \frac{x^3}{(1-x)^2}, \quad \sum_{n=0}^{\infty} x^{n+2} = \frac{x^2}{1-x}.$$

Thus, we get the following equation on $f(x)$:

$$f(x) - 2 - 4x = 3x f(x) - 6x + 4x^2 f(x) - \frac{6x^3}{(1-x)^2} - \frac{x^2}{1-x},$$

which gives

$$f(x) = \frac{2 - 6x + 5x^2 - 7x^3}{(1-x)^2(1-3x-4x^2)}$$

Expressing $f(x)$ as a sum of partial fractions gives

$$f(x) = \frac{1}{(1-x)^2} - \frac{1}{1-x} + \frac{1}{1+x} + \frac{1}{1-4x}.$$

This immediately gives the answer:

$$f(x) = \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} 4^n x^n,$$

so

$$a_n = n + (-1)^n + 4^n.$$

Problem 5. Compute the sum: $\sum_{n=1}^{\infty} \frac{10^n + 3n}{n!}$ (Advice: note the range of summation.)

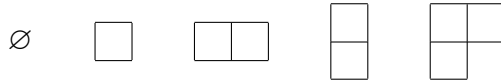
Solution 5. We have

$$\sum_{n=1}^{\infty} \frac{10^n + 3n}{n!} = \sum_{n=1}^{\infty} \frac{10^n}{n!} + 3 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \left(\sum_{n=0}^{\infty} \frac{10^n}{n!} - 1 \right) + 3 \sum_{n=0}^{\infty} \frac{1}{n!} = e^{10} - 1 + 3e.$$

2. CATALAN NUMBERS

Of these two problems, solve at least one (of your choice)

Problem 6. Show that the number of Young diagrams which are inside the staircase shaped Young diagram $(n-1, n-2, \dots, 1)$ is Catalan_n . (Hint: bijection with certain up-right paths.)



Solution 6. Bijection with up-right paths on the lattice starting at $(0,0)$ and ending at (n,n) which never go below the diagonal — the part of the $n \times n$ square which is above such paths is a Young diagram:

Problem 7. Show that the number of tiling of the staircase shape $(n, n-1, n-2, \dots, 2, 1)$ with n rectangles is Catalan_n . (Hint: derive the Catalan recurrence relation by deleting one of the tiling rectangles.)

Solution 7. Deleting the upper left rectangle leads to two tilings of smaller staircase Young diagrams, which gives the desired recurrence. Indeed, let the upper left rectangle be $K \times L$. It is readily seen that $K + L = n + 1$, and deleting this rectangle we get a pair of independent tilings of the staircase shapes $(K-1, K-2, \dots, 2, 1)$ and $(L-1, L-2, \dots, 2, 1)$. Thus, we get our recurrence for the unknown number of tilings:

$$C_n = \sum_{K,L: K+L=n+1} C_{K-1} C_{L-1} = \sum_{K=1}^n C_{K-1} C_{n-K} = \sum_{K=0}^{n-1} C_K C_{n-K-1},$$

which is the Catalan recurrence. Noting that $C_0 = C_1 = 1$, we conclude the proof.

3. LAGRANGE INVERSION

Problem 8. Write a solution to the cubic equation $x - 1 = ax^3$, that is, find the generating series $x = x(a)$.

Solution 8. Set $y = x - 1$, then we have the equation $y = a(y + 1)^3$, and by Lagrange inversion,

$$y(a) = y_1 a + y_2 a^2 + y_3 a^3 + \dots,$$

and

$$y_n = \frac{1}{n} [u^{n-1}] (y - 1)^{3n} = \frac{1}{n} \binom{3n}{n-1},$$

so

$$y(a) = \sum_{n=1}^{\infty} \frac{1}{n} \binom{3n}{n-1} a^n,$$

and the answer is

$$x(a) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \binom{3n}{n-1} a^n.$$

4. ASYMPTOTICS

Problem 9. Let a_n be a sequence with the generating function

$$f(x) := \sum_{n=0}^{\infty} a_n x^n = \frac{2}{(1-x)(1-2x)(1-3x)}.$$

- (a) What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$? (Hint: use singular points.)
- (b) Find minimal possible R such that a_n grows slower than $(\frac{1}{R} + \epsilon)^n$ for any $\epsilon > 0$.
- (c) Expand $f(x)$ as a sum of partial fractions. Find an explicit formula for a_n .

Solution 9. (a) The singular points are, clearly, $x = 1$, $x = 1/2$ and $x = 1/3$. So the closest one is $x = 1/3$, and the radius of convergence is the distance to this closest point, it is $R = \frac{1}{3}$.

(b) The same $R = \frac{1}{3}$ serves here, so a_n grows slower than $(3 + \epsilon)^n$ for any $\epsilon > 0$.

(c) The expansion of $f(x)$ is

$$f(x) = \frac{1}{1-x} - \frac{8}{1-2x} + \frac{9}{1-3x} = \sum_{n=0}^{\infty} [1 - 8 \cdot 2^n + 9 \cdot 3^n] x^n.$$

so

$$a_n = 1 - 8 \cdot 2^n + 9 \cdot 3^n.$$

Problem 10. Let $a_n = \binom{4n}{2n}$.

(a) Make sure that the sequence a_n is hypergeometric. Deduce hypergeometric-type asymptotics for the sequence a_n .

(b) Using Stirling's formula, find *exact* asymptotics of the sequence a_n .

Solution 10. First, we write asymptotics of $b_k = \binom{2k}{k}$, and then substitute $k = 2n$ everywhere.

(a) We have

$$\frac{b_{k+1}}{b_k} = \frac{\binom{2k+2}{k+1}}{\binom{2k}{k}} = \frac{(2k+2)(2k+1)}{(k+1)^2} = 4 \frac{k^2 + \frac{3}{2}k + \frac{1}{2}}{k^2 + 2k + 1},$$

so we have $b_k \sim \text{Const} \cdot 4^k \cdot k^{-\frac{1}{2}}$, and thus $a_n \sim \text{Const}' \cdot 4^{2n} \cdot n^{-\frac{1}{2}}$ (with some other constant).

(b) Using Stirling's formula,

$$\binom{2k}{k} \sim \frac{(2k)!}{k!k!} = \frac{\sqrt{2\pi \cdot 2k}}{\sqrt{2\pi \cdot k} \sqrt{2\pi \cdot k}} \frac{(2k/e)^{2k}}{(k/e)^{2k}} = \sqrt{\frac{1}{\pi k}} \cdot 2^{2k},$$

and thus

$$a_n \sim \sqrt{\frac{1}{2\pi n}} \cdot 2^{4n}.$$

5. SUPPLEMENTARY PROBLEMS

Problem 11. Give a *combinatorial* proof of (1) in Problem 2. (Hint: interpret 3^n as the number of all sequences of length n with letters a, b, c .)

Solution 11. On the right we have 3^n , which is the number of all sequences of length n with letters a, b, c . Any such sequence has, say, $n - k$ letters a , and k remaining letters are b or c . For any $k = 0, 1, \dots, n$, the number of ways to choose such a sequence with k letters a is $2^k \binom{n}{k}$: here $\binom{n}{k} = \binom{n}{n-k}$ means the number of ways to choose $n - k$ positions to place a 's, and 2^k is the number of words of length k with letters b, c .

Problem 12. Solve the differential equation using generating functions:

$$f'(x) + xf(x) = 0.$$

(Advice: substitute your answer into the equation to check yourself.)

Solution 12. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then we get the equation

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

or, equivalently,

$$a_1 + \sum_{n=1}^{\infty} \left[(n+1)a_{n+1} + a_{n-1} \right] x^n = 0,$$

which means that (we compare the coefficients by $1, x, x^2, x^3, \dots$):

$$a_1 = 0, \quad a_2 = -\frac{a_0}{2}, \quad a_3 = 0, \quad a_4 = -\frac{a_2}{4} = \frac{a_0}{8}, \quad a_5 = 0, \quad a_6 = -\frac{a_4}{6} = -\frac{a_0}{48}, \quad \dots$$

We see that the odd coefficients a_n are zero, and for the even coefficients we have

$$a_{2k} = -\frac{a_{2k-2}}{2k},$$

which readily implies that

$$a_{2k} = \frac{(-1)^k 2^{-k}}{k!} a_0.$$

It remains to sum the power series:

$$f(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-k}}{k!} x^{2k} = a_0 e^{-\frac{x^2}{2}}.$$

Problem 13. Show that any *minor* (= determinant formed by some of the rows i_1, \dots, i_k and the same number of columns j_1, \dots, j_k , k is arbitrary) of the matrix

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{N-1} & \alpha^N \\ 0 & 1 & \alpha & \dots & \alpha^{N-2} & \alpha^{N-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

is nonnegative (here $\alpha > 0$). (Hint: use nonintersecting paths approach.)

Solution 13. Draw a graph, and observe that any minor corresponds to a choice of entrances/exits. So its determinant is nonnegative because it is (by KMGLGV) the “number” of collections of nonintersecting paths (i.e., every path is counted with its weight; the weight of a collection = product of weights of paths).