Nonequilibrium particle systems in inhomogeneous space

Leonid Petrov

University of Virginia • MIT • IITP

March 5, 2018

(part is joint work in progress with Alisa Knizel and Axel Saenz;
also based on earlier work with Alexei Borodin, Alexey Bufetov, and Daniel Orr)
Outline

KPZ equation and TASEP

Inhomogeneous TASEP and slow bond problem

Integrable particle systems in inhomogeneous space

PushTASEP

Schur measures

Continuous space TASEP

Conclusions
**KPZ equation and TASEP**

KPZ equation [Kardar, Parisi, and Zhang, 1986] — a stochastic PDE model for randomly growing interface $h(t, x), t > 0, x \in \mathbb{R}$:

\[
\frac{\partial h(t, x)}{\partial t} = \frac{\partial^2 h(t, x)}{\partial x^2} + \left( \frac{\partial h(t, x)}{\partial x} \right)^2 + \eta(t, x), \quad \mathbb{E}\eta(t, x)\eta(t', x') = \delta(t - t')\delta(x - x')
\]

(the time evolution of the interface is governed by the smoothing and the slope-dependent growth terms, plus random noise)

- Existence and uniqueness of solutions to this ill-posed equation [Hairer, 2014]

- Approximation of solutions of the KPZ equation by discrete-space interacting particle systems such as weakly ASEP [Bertini and Giacomin, 1997], etc.

- Exact distributions and limits (e.g. $t \rightarrow +\infty$) of $h(t, x)$ for specific and (conjecturally) general initial data $h(0, x)$ [Amir, Corwin, and Quastel, 2011, Matetski, Quastel, and Remenik, 2017], etc.
surface growth model

liquid crystal experiment
[Takeuchi and Sano, 2010]

simulation by M. Hairer

Leonid Petrov ● Nonequilibrium particle systems in inhomogeneous space
TASEP (totally asymmetric simple exclusion process)

Each particle has an exponential clock with rate 1: \( P(\text{wait} > s) = e^{-s}, \ s > 0 \), clocks are independent for each particle. When the clock rings, the particle jumps to the right by one if the destination is not occupied.

Theorem ([Johansson, 2000])
Start TASEP from the step initial configuration \( x_i(0) = -i, \ i = 1, 2, \ldots \). Let \( h(t, x) \) be the number of particles \( \geq x \) at time \( t \). Then

\[
\lim_{L \to +\infty} P\left( \frac{h(\tau L, \chi L) - L \bar{h}(\tau, \chi)}{c \chi L^{1/3}} \geq -s \right) = F_{\text{GUE}}(s),
\]

where \( F_{\text{GUE}} \) is the GUE (Gaussian Unitary Ensemble) Tracy–Widom distribution originated in random matrix theory [Tracy and Widom, 1993]
General principle / conjecture: models in KPZ class (including the KPZ equation) at large times and scales behave as TASEP at large times and scales

Starting from Johansson’s theorem, there is a very good understanding of TASEP asymptotics:

- multipoint distributions
- particle-dependent speeds
- other initial conditions including general ones
- extensions to other models such as ASEP


- Some important aspects are missing:

The focus today is on asymptotics of particle systems in inhomogeneous space

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
Inhomogeneous TASEP and slow bond problem

— one of the most complicated aspects of TASEP asymptotics, still not fully understood
A stochastic particle system is called (space) *inhomogeneous* if the speed/jump rate of a particle depends on its location. Particles themselves are *identical*.

**Example: inhomogeneous TASEP**

![Diagram of inhomogeneous TASEP]

**Slow bond problem.** Let $\xi_i = 1$ for $i \neq -1$, and $\xi_{-1} = 1 - \varepsilon \in [0, 1]$.

**Question:** For step IC, does the flux of particles through zero decrease (from $\frac{1}{4}$ for $\varepsilon = 0$) for any $\varepsilon > 0$? Or is there a critical value $\varepsilon_c \neq 0$?

This question received competing predictions from various groups of physicists [Janowsky and Lebowitz, 1992, Costin, Lebowitz, Speer, and Troiani, 2013].

This is a *hard analytic problem*: integrable tools break down for inhomogeneous TASEP.
Warm up: hydrodynamics (why current through 0 is \( \frac{1}{4} \))

*Law of large numbers* for regular (locally constant) behavior \( \xi_i = \xi(i/t) \) and \( t \to \infty \).

Understanding translation invariant stationary distributions in a *homogeneous* system, write down a PDE for the limiting density in the inhomogeneous case

**Theorem (Liggett)**

Bernoulli measures are all non-trivial extremal stationary measures of TASEP.

Let \( \rho \) be the density of the Bernoulli measure, then the flux (current) is \( j(\rho) = \rho(1 - \rho) \).

The continuity equation for the limiting density \( \rho(t, x) \) (if it exists) is

\[
\frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} \left( \xi(x) \rho(t, x) (1 - \rho(t, x)) \right) = 0, \quad \rho(0, x) = \rho_0(x).
\]

For \( \xi \equiv 1 \), \( \rho_0(x) = \mathbf{1}_{x<0} \), we have \( \rho(t, x) = \frac{1}{2} (1 - x/t), \ |x| < t \). So \( \rho(t, 0) = \frac{1}{2} \).

[GKS10]: formulas when \( \xi \) takes 2 values. Conjecturally, Tracy–Widom fluctuations

Slow bond problem and last passage percolation

For slow bond problem, \( \xi_i \) is not locally constant at 0. The slow bond problem was recently solved in [Basu, Sidoravicius, and Sly, 2014]; invariant measures described in [Basu, Sarkar, and Sly, 2017]:

For every \( \varepsilon > 0 \), the asymptotic current through zero is strictly less than \( \frac{1}{4} \).

The proof uses mapping to last passage percolation with \( \exp(1 - \varepsilon) \) random variables on the diagonal. Connection to algebraic structures (e.g. Schur measures) unknown.

- TASEP on \( \mathbb{Z} \)
- directed last passage percolation
- corner growth
- longest increasing subsequences
- tandem queues
Integrable particle systems in inhomogeneous space

— let us “repair” the inhomogeneous TASEP by taking a “similar” system which is exactly solvable
At least two close relatives of TASEP are integrable in inhomogeneous space:

- **PushTASEP**: a known old relative of TASEP, also called long-range TASEP, a special case of the Toom’s model [Derrida, Lebowitz, Speer, and Spohn, 1991], known since 1970s

- **Continuous space TASEP**: a new model which is a $q = 0$ version of the exponential jump model [Borodin and Petrov, 2017]

The ultimate triggers of integrability: Yang-Baxter equation and Schur measures

Application of Yang-Baxter equation in probabilistic context: [Borodin and Petrov, 2016]

Integrable deformations (TASEP → ASEP style):

- PushTASEP → stochastic six vertex model [Borodin, Corwin, and Gorin, 2016b] (Hall-Littlewood structure);

- Continuous space TASEP → exponential jump model ($q$-Whittaker structure)
• Each particle at location \( x \) has an independent exponential clock with rate \( \xi_x \).

• When the clock rings, the particle jumps to the right by one, pushing the packed cluster on the right also by one.

• **Step IC:** initially particles occupy 1, 2, 3, \ldots At \( t = 0^+ \): infinitely many jumps

• Height function \( h_{\text{push}}(t, N) := \#\{\text{particles} \leq N \text{ at time } t\} \).

  Initially \( h_{\text{push}}(0, N) = N \).

---

**Theorem (column RSK; or \( t = 0 \) in [Borodin, Bufetov, and Wheeler, 2016a])**

\[ h_{\text{push}}(t, N) \overset{d}{=} N - \lambda'_1, \text{ where } \lambda \text{ is distr. as Schur measure } \propto s_\lambda(\xi_1, \ldots, \xi_N)s_\lambda(\rho_t). \]

\((N - \lambda'_1 \text{ is the number of zero parts in the partition } \lambda_1 \geq \ldots \geq \lambda_N \geq 0)\)

(Schur measures will be discussed a little later)
Remark: PushTASEP and TASEP via Schur measures

There are three other results connecting PushTASEP and TASEP with Schur measures:

- [row RSK, BF] Let $x_i$ be the $i$th particle of PushTASEP with step IC and speed $a_i$, then $x_i(t) \overset{d}{=} \lambda_1 + i$, where $\lambda \sim s_\lambda(a_1, \ldots, a_i)s_\lambda(\rho_t)$;

- [column RSK, BF] Let $y_i$ be the $i$th particle of TASEP with step IC and speed $a_i$, then $y_i(t) \overset{d}{=} \lambda_i - i$, where $\lambda \sim s_\lambda(a_1, \ldots, a_i)s_\lambda(\rho_t)$;

- [via LPP and row RSK] The height function of TASEP with step IC is related to the last passage percolation times with exponential weights. These LPP times are expressed (via row RSK) through certain simple limits of Schur measures.

Neither of these well-known correspondences addresses particle systems in inhomogeneous space

(the LPP is used in recent TASEP slow bond works, but the connection to Schur measures breaks)
PushTASEP limit shape

The limit regime is $t = \tau L$, $x = \chi L$, $\xi_i = \xi(i/L)$ for a sufficiently nice $\xi(\cdot)$, and $L \to +\infty$.

Theorem (P., in progress)
Under mild assumptions on $\xi(\cdot)$, there exists a limit shape:

$$\frac{1}{L} h_{\text{push}}(\tau L, \chi L) \to h_{\text{push}}(\tau, \chi), \quad L \to +\infty.$$ 

Hydrodynamics — a priori PDE for limiting density

For PushTASEP, translation invariant stationary measures are Bernoulli [Guiol, 1997, Andjel and Guiol, 2005]. For $\xi_i \equiv 1$, the flux is $j(\rho) = \frac{\rho}{1 - \rho}$

The PDE for the limiting density $\rho(\tau, \chi) = \frac{\partial}{\partial \chi} h_{\text{push}}(\tau, \chi)$ is

$$\frac{\partial}{\partial \tau} \rho(\tau, \chi) + \frac{\partial}{\partial \chi} \left( \frac{\xi(\chi) \rho(\tau, \chi)}{1 - \rho(\tau, \chi)} \right) = 0, \quad \rho(0, \chi) = \rho_0(\chi).$$

(step IC: $\rho_0(\chi) = 1_{\chi > 0}$)
Limit shape, explicit formulas

Step IC: $\rho_0(\chi) = 1_{\chi > 0}$.

The curved part of the limiting density admits the parametrization

$$\rho(\tau, \chi) = \frac{z(\tau, \chi)}{z(\tau, \chi) - \xi(\chi)}, \text{ where } \tau = \int_0^\chi \frac{\xi(y)}{(z(\tau, \chi) - \xi(y))^2} dy, \quad z \in (-\infty, 0).$$

Example: $\xi \equiv 1$, then the limit shape is quadratic (horizontal axis is $\chi$, the scaled location):
More examples of the density when we insert a macroscopic slowdown or speedup:
Simulations of the height function $h_{push}$ (http://lpetrov.cc/simulations)

Below: linear shifts to better see discontinuity of derivatives of the height function
PushTASEP fluctuations

Let $\tau_e(\chi) := \int_0^\chi \frac{dy}{\xi(y)}$, this is the scaled time when the first particle reaches $\chi$. So, $\rho(\tau, \chi) \equiv 0$ for $\tau > \tau_e(\chi)$.

**Theorem (P., in progress ’18)**

For any $(\tau, \chi)$ with $\tau < \tau_e(\chi)$, we have

$$
\lim_{L \to +\infty} \mathbb{P} \left( \frac{h_{\text{push}}(\tau L, \chi L) - L h_{\text{push}}(\tau, \chi)}{d(\tau, \chi) L^{1/3}} \geq -s \right) = F_{\text{GUE}}(s),
$$

where $F_{\text{GUE}}$ is the GUE Tracy–Widom distribution and

$$
d(\tau, \chi) = \left( \int_0^\eta \frac{z(\tau, \chi)^2 \xi(y)}{(\xi(y) - z(\tau, \chi))^3} dy \right)^{1/3}
$$
Slow bonds

Q: Insert a microscopic or mesoscopic slowdown or speedup? Consider slowdowns.

Unlike TASEP, particles can be pushed through slowdowns. Therefore, a single slow bond does not affect the global behavior. Let us consider slow zones of size $\varepsilon \ll L$.

- If $\varepsilon \ll L^{1/3}$, it does not affect fluctuations.
- If $\varepsilon \sim L^{1/3}$, it inserts a deterministic shift $\frac{h - L\eta}{\sigma L^{1/3}} \to X_{GUE} + c$.
- If $\varepsilon \gg L^{1/3}$ all the way to $\sim L^{2/3}$, it inserts a growing shift as in $\frac{h - L\eta + cL^{\gamma}}{L^{1/3}}$, with $1/3 < \gamma = \gamma(\varepsilon) \leq 2/3$. 

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
Schur measures

— main integrability tool
Schur measures [Okounkov, 2001] — a powerful tool in integrable probability, in particular used to solve inhomogeneous PushTASEP

### Definition (Schur polynomials)

\[ \lambda = (\lambda_1 \geq \ldots \lambda_N \geq 0), \quad s_\lambda(\xi_1, \ldots, \xi_N) := \frac{\det[\xi_i^{\lambda_j + N - j}]_{i,j=1}^N}{\det[\xi_i^{N-j}]_{i,j=1}^N} \]

\[ s_\lambda(\xi_1, \ldots, \xi_N) \geq 0 \text{ if all } \xi_i \geq 0. \]

### Definition (Plancherel specialization)

\[ s_\lambda(\rho_t) := \lim_{K \to +\infty} s_\lambda \left( \frac{t}{K}, \ldots, \frac{t}{K} \right) (K \text{ times}) \]

### Definition (Schur measure)

\[ \mathbb{P}(\lambda) := \frac{1}{Z} s_\lambda(\xi_1, \ldots, \xi_N)s_\lambda(\eta_1, \ldots, \eta_N), \quad Z = \prod_{i,j=1}^{N} \frac{1}{1 - \xi_i \eta_j}. \]

Many parameters \( \xi_i, \eta_j \)

Explicit normalization follows from the Cauchy identity.
Working example (related to PushTASEP)

Of particular interest is the Schur measure \( \propto s_{\lambda}(\xi_1, \ldots, \xi_N)s_{\lambda}(\rho_t) \)

When \( \xi_i \equiv 1 \), it is sometimes called the Schur-Weyl measure, and appears in dimension counting in Schur-Weyl duality.

Schur-Weyl measure is closely related to the Plancherel measure on partitions and longest increasing subsequences

[Baik, Deift, and Johansson, 1999], [Okounkov, 2000], [Borodin, Okounkov, and Olshanski, 2000], [Biane, 2001], [Romik, 2015], etc.

Determinantal structure

The half-infinite random point configuration \( \{\lambda_j - j\}_{j \geq 0} \) is a determinantal point process on \( \mathbb{Z} \), that is,

\[
P(\text{random configuration } \{\lambda_j - j\} \text{ contains } a_1, \ldots, a_r) = \det_{i,j=1}^{r} \left[ K(a_i, a_j) \right].
\]
The kernel for $\propto s_\lambda(\xi_1, \ldots, \xi_N)s_\lambda(\rho_t)$ is

$$K(x, y) = \frac{1}{(2\pi i)^2} \oint \oint \frac{dwdz}{z-w} \frac{w^y}{z^{x+1}} e^{t(z-w)} \prod_{i=1}^{N} \frac{1 - \xi_i/z}{1 - \xi_i/w}$$

(Contours are around 0 and $\{\xi_i\}$, and $|z| > |w|$)

- Asymptotic questions about Schur measures can in principle be answered via asymptotic analysis of contour integrals (*saddle point methods*)

- Extension to Schur processes [*Okounkov and Reshetikhin, 2003*]

- Markov dynamics on Schur processes (changing their specializations) are a rich source of integrable particle systems in 1 and 2 dimensions.

**Remark.** Markov dynamics on Schur processes are related / extend to

1. *shuffling* for lozenge or domino tilings;
2. Robinson-Schensted-Knuth insertion;
3. integrable models of random polymers in random media; etc.
(A sample of) models solvable by Schur measures

- Homogeneous or particle-inhomogeneous TASEP on $\mathbb{Z}$
- directed last passage percolation
- corner growth
- longest increasing subsequences
- tandem queues

Plane partitions and other random tilings (noncolliding walks; dimer models; etc.)

Tiles or particles along certain cross-sections are distributed as Schur measures

Also: random matrix type models, $z$-measures, polynuclear growth, ...

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
Continuous space TASEP
A continuous time particle system $X(t)$ on ordered particles $x_1 \geq x_2 \geq \ldots$ in $\mathbb{R}_{\geq 0}$.

Step IC: initially infinitely many particles at 0, the rest of the space is empty.

- one particle can leave a stack at location $x$ at rate $\zeta(x)$, where $\zeta$ is an arbitrary positive piecewise continuous speed function;
- the jumping particle wants to jump an exponential distance with mean $1/L$;
- particles preserve order — an overflying particle joins the first stack to the right

Height function $h_{\text{cont}}(t, x) := \# \{\text{number of particles } \geq x \text{ at time } t\}$.

Limit regime: $L \to +\infty$, $t = L\tau$ — more particles, long time, short jumps; the speed function $\zeta(\cdot)$ and location $x$ are not scaled.
• This is a “natural” definition of TASEP in continuous space

• Also a continuous space limit of the discrete space generalized TASEP of [Derbyshev, Povolotsky, and Priezzhev, 2015]

• Resembles queuing systems, with Poisson service

• Can incorporate roadblocks (= slow bonds) in the space catching particles with fixed probability; for simplicity let’s leave this for now

• A $q$-deformation was first studied in [Borodin and Petrov, 2017]. Continuous space TASEP is the $q = 0$ limit, and methods of [BP17] break for $q = 0$

**Theorem ([Orr and Petrov, 2017])**

Take $\lambda_M$ under $\propto s_\lambda \left( \zeta \left( \frac{x}{M} \right), \zeta \left( \frac{2x}{M} \right), \ldots, \zeta(x) \right) s_\lambda \left( \frac{e^{-L/M}}{\zeta(x/M)}, \frac{e^{-L/M}}{\zeta(2x/M)}, \ldots, \frac{e^{-L/M}}{\zeta(x)}; \rho_t \right)$.

Then $h_{\text{cont}}(t, x) \overset{d}{=} \lim_{M \to \infty} \lambda_M$.

(recall: $L^{-1}$ is the mean jump distance in the continuous TASEP)
Remark: Invariance under permutations / exchangeability

The height function’s distribution in both PushTASEP and continuous space TASEP is invariant under permutations of space inhomogeneity:

\[ h_{\text{push}}(t, x) \sim s_\lambda(\xi_1, \ldots, \xi_x)s_\lambda(\rho_t), \]
\[ h_{\text{cont}}(t, x) \sim s_\lambda \left( \frac{1}{\zeta([0, x])} \right) s_\lambda \left( \zeta([0, x]) \right); \rho_t \]

(Schur polynomials are symmetric)

These posterior facts can be traced to the Yang-Baxter equation for the \( sl_2 \) higher spin six vertex model

This invariance under permutations of the environment can be viewed as a good indicator towards solvability of space-inhomogeneous models

(Does not naively work for TASEP with a slow bond)
Simulations of continuous space TASEP height function

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
Simulations of continuous space TASEP height function
Simulations of continuous space TASEP height function
Simulations of continuous space TASEP height function
Simulations of continuous space TASEP height function
Simulations of continuous space TASEP height function
Simulations of continuous space TASEP height function
Hydrodynamics in continuous space TASEP

Examples of translation invariant stationary distributions are Poisson processes on $\mathbb{R}$ with random geometric number of particles at points of Poisson process.

The flux (current) is $j(\rho) = \frac{2\rho + 1 - \sqrt{4\rho + 1}}{2\rho}$ (why generating function for Catalan numbers?), and the PDEs for the limiting density and the limiting height function $h(\tau, x)$ are

$$\frac{\partial}{\partial \tau} \rho(\tau, x) + \frac{\partial}{\partial x} \left( \zeta(x) j(\rho(\tau, x)) \right) = 0 \quad \Rightarrow \quad h_x(\tau, x) = -\frac{\zeta(x) h_\tau(\tau, x)}{(\zeta(x) - h_\tau(\tau, x))^2}$$
Fix piecewise continuous $\zeta(\cdot)$, scale time $t = \tau L$, send $L \to +\infty$

**Theorem ([Knizel, Petrov, and Saenz, 2018], in preparation)**
There exists almost sure limit shape $h(\tau, x) = \lim_{L \to \infty} \frac{1}{L} h_{cont}(\tau, x)$.

**Edge:** $h(\tau, x) \equiv 0$, $x \geq x_e(\tau)$, where $\tau = \int_0^{x_e(\tau)} \frac{dy}{\zeta(y)}$. For $x \in (0, x_e(\tau))$:

- Let $w_x$ be the unique solution of $\tau w = \int_0^x \frac{w\zeta(y)(w + \zeta(y))}{(\zeta(y) - w)^3} dy$ on the interval $0 < w_x < \min_{0 \leq y < x} \zeta(y)$. If $\zeta$ jumps down, the interval shrinks and $w_x$ is discontinuous in $x$.

- The limit shape is $h(\tau, x) = \tau w_x - \int_0^x \frac{\zeta(y)w_x}{\zeta(y) - w_x} dy$. One can check that it satisfies the continuity equation.

**Example:** If $\zeta(x) = 1_{x \leq x_0} + b1_{x > x_0}$, the limit shape is piecewise degrees 3 and 6.
Fluctuations for continuous TASEP

Let $\zeta(\cdot)$ be piecewise continuous, and continuous at 0.

**Theorem**

When $x \in (0, x_e(\tau))$, define $\sigma_x := 2^{-\frac{1}{3}} \left[ \int_0^x \frac{2(w_x)^2 \zeta(y)(w_x + 2\zeta(y))}{(\zeta(y) - w_x)^4} dy \right]^{\frac{1}{3}} > 0$.

Then

$$\lim_{L \to \infty} \mathbb{P} \left( \frac{h_{\text{cont}}(\tau L, x) - L h(\tau, x)}{L^{1/3} \sigma_x} \geq -r \right) = F_{\text{GUE}}(r), \quad r \in \mathbb{R}. $$

**Note:** $F_{\text{GUE}}$ fluctuations hold even at the points of infinite traffic jams

• There is a phase transition and a critical value $\zeta_c(\tau)$ of the slowdown speed $\zeta_c$: for speed $\zeta > \zeta_c$ the height function is continuous, and for $\zeta < \zeta_c$ the height function becomes discontinuous.

• The density is always discontinuous. In infinite traffic jam the density is infinite, too.
Fluctuations at a traffic jam

- $\xi = 1_{x<b} + \xi \cdot 1_{x\geq b}, \, \xi < 1$
- There exists $\tau_c$ s.t. $h(\tau, x)$ is discontinuous for $\tau > \tau_c$ and continuous for $\tau < \tau_c$
- For all finite $\varepsilon$,
  \[
  \frac{h(\tau_c, b + \varepsilon) - L h(\tau_c, b + \varepsilon)}{\sigma_{b+\varepsilon}(\tau_c) L^{1/3}} \to F_{GUE} \quad (*)
  \]
  ($\sigma_{b+\varepsilon}(\tau)$ also discontinuous for $\tau > \tau_c$)

Theorem (Fluctuation transition around a traffic jam)
For any small $c > 0$:
- For $\varepsilon \ll L^{-4/3-c}$, (*) holds with $L h(\tau_c, b)$ and $F_{GUE}$
- For $\varepsilon = \delta L^{-4/3}$, (*) holds with $L h(\tau_c, b)$ and $F_{GUE}^{(\delta)}$, a deformation
- For $\varepsilon \gg L^{-4/3+c}$, (*) holds with $L h(\tau_c, b + \varepsilon)$ (which differs from $L h(\tau_c, b)$ by more than $L^{1/3}$) and $F_{GUE}$

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
The deformation $F_{GUE}^{(\delta)}$ is one of the deformations given in [Borodin and Peche, 2008], and is a Fredholm determinant of a deformation of the Airy kernel:

$$K^{(\delta)}(r, r') = \frac{1}{(2\pi i)^2} \int_{e^{-2\pi i/3\infty}}^{e^{2\pi i/3\infty}} dw \int_{e^{-\pi i/3\infty}}^{e^{\pi i/3\infty}} dz \frac{1}{z - w} \times \exp \left\{ \frac{z^3}{3} - \frac{w^3}{3} - zr + wr' - \frac{\delta}{z} + \frac{\delta}{w} \right\}$$
Roadblocks / slow bonds in continuous TASEP

In continuous space TASEP, add special locations $\sigma_i$ called roadblocks (or slow bonds).

In these locations, every particle flying over can be stopped with probability $p \in (0, 1)$. When there is a particle at the roadblock, it will capture all flying particles automatically.

The rate of particles $\zeta(\sigma_i)$ leaving a roadblock is fixed.

- With roadblocks the system is still integrable
- Roadblock behavior does not depend on $p \in (0, 1)$
- There is a critical value $\zeta_c$ of $\zeta(\sigma_i)$; above it the roadblock does not affect the limit shape, and below it the limit shape is affected.
- If the limit shape is affected, the fluctuations are at scale $L^{1/2}$ and Gaussian. The phase transition is of [Baik, Ben Arous, and Péché, 2005] type.
Examples

(Here there are also Gaussian fluctuations at scale $\sqrt{L}$ because $\xi(0) < \xi(0^+)$)

Leonid Petrov  •  Nonequilibrium particle systems in inhomogeneous space
Traffic jams and slow bonds in continuous TASEP

Leonid Petrov • Nonequilibrium particle systems in inhomogeneous space
Conclusions

- The original slow bond problem for TASEP is hard and so far resists integrable tools

- There are two related models in inhomogeneous space, exactly solvable through Schur measures; both allow to generate slow bond type behavior

- Common feature — invariance under permutations of the environment

- Explicit limit shape formulas and fluctuation results

- Slow bonds or zones in these models can be investigated in detail:
  - in continuous space TASEP there is a new type of phase transitions — infinite traffic jams; for them there is a critical value of the macroscopic slowdown speed
  - in continuous space TASEP there are also critical values of the slowdown speed at roadblocks (= single slow bonds)
  - in PushTASEP, need mesoscopic or macroscopic size of the slowdown to affect limit shape or fluctuations; exact results are in progress
...can slow bond be “repaired” better?...
...can slow bond be “repaired” better?...

Thank you!
References


